From phylogenetics to statistical physics: tensor wiring diagrams and the applications of tensors to the sciences

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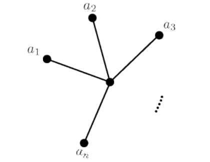
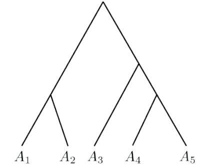
MAA MathFest, July 2017

The end goal... applications of tensor networks

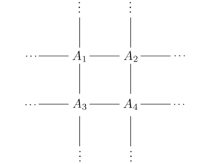
Algebraic statistics



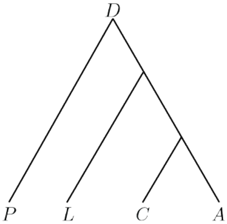
Phylogenetics



Physics

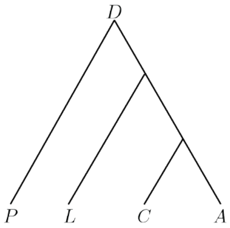


Phylogenetics



We suspect that pigeons, lizards, crocodiles, and alligators evolved from dinosaurs in the manner above... How can we check?

Phylogenetics



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We associate a vector space to each vertex on the graph. In this case, each vector space is $\mathbb{R}^4$ because there are 4 DNA bases: A, T, C, and G.

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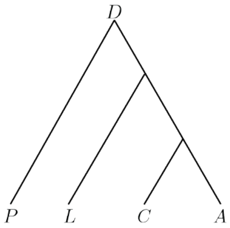

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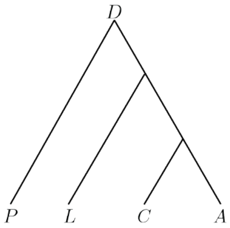
We associate a vector space to each vertex on the graph. In this case, each vector space is $\mathbb{R}^4$ because there are 4 DNA bases: A, T, C, and G. Every living species has a certain percentage of A, T, C, and G. That is, every species can be represented by a vector in $\mathbb{R}^4$ whose components add up to 1.

Phylogenetics



We check if:

$$P \otimes L \otimes C \otimes A \in TNS(G, \vec{w}, (\mathbb{R}^4)^{\otimes 7}) \downarrow_{(\mathbb{R}^4)^{\otimes 4}} \subset (\mathbb{R}^4)^{\otimes 4}$$

More formally, we check,

$$\begin{aligned}[P \otimes L \otimes C \otimes A] &\in \sigma_4(Seg(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}(\mathbb{R}^4 \otimes \mathbb{R}^4))) \\ &\cap \sigma_4(Seg(\mathbb{P}(\mathbb{R}^4 \otimes \mathbb{R}^4) \times \mathbb{P}^3 \times \mathbb{P}^3))\end{aligned}$$

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The dual space

Definition 1

The **dual space** to a vector space, V, is denoted V^* and is defined as:

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Remark

If elements of V are represented by column vectors, then elements of V^* are row vectors, and the map $u \mapsto \beta(u)$ is given by row-column matrix multiplication. That is,

$$\beta(u) = (\beta_1 \cdots \beta_n) \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} = \beta_1 u^1 + \cdots + \beta_n u^n$$

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Multilinear maps are functions

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Definition 2

The space of such multilinear functions is denoted $V_1^* \otimes \cdots \otimes V_k^*$, and elements $T \in V_1^* \otimes \cdots \otimes V_k^*$ are called **tensors**.

Linear maps as tensors

Linear maps can be realized as tensors! More precisely, there is an isomorphism between linear maps and tensors.

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Conversely, if $\tilde{L} : V \times W \to \mathbb{C}$ is a given tensor, the corresponding linear map is $L : v \mapsto \tilde{L}(v, \cdot)$ where we partially satisfy the tensor such that $L(v) \in (W^*)^*$. In fact, $(W^*)^* = W$.

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Because of how natural this isomorphism is, we just say $L = \tilde{L}$.
In general, any multilinear map $T : V_1 \otimes \cdots \otimes V_s \to W_1 \otimes \cdots \otimes W_k$ can be realized as $T : V_1 \otimes \cdots \otimes V_s \otimes W_1^* \otimes \cdots \otimes W_k^* \to \mathbb{C}$.

Linear maps as tensors

The reasoning behind this realization is that if

$$L = \sum_i \alpha^i \otimes w_i$$

where $\alpha^i \in V^*$ and $w_i \in W$. Then, we can say

$$L(v) = \sum_i \alpha^i(v) w_i \in W$$

or

$$L(v, \beta) = \sum_i \alpha^i(v) \beta(w_i) \in \mathbb{C}$$

One tensor, many realizations

Pick a basis v_i for V, which induces a basis, α^i for V^*. Consider $Id \in V^* \otimes V$,

$$Id = \alpha^1 \otimes v_1 + \cdots + \alpha^n \otimes v_n = \sum_s \alpha^s \otimes v_s$$

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Id can be thought of as a map from $V \times V^* \to \mathbb{C}$, or a map from $V \to V$. As we saw in the previous slide, Id is a matrix, so if we feed it a vector $u = (u^1, \cdots, u^n)^\intercal \in V$.

$$Id(u) = \alpha^1(u)v_1 + \cdots + \alpha^n(u)v_n = u^1 v_1 + \cdots + u^n v_n = u$$

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Or, if we feed it another matrix, say $B = \sum b^i_j v_i \otimes \alpha^j$, then:

$$Id(B) = \sum b^i_j \delta^s_i \delta^j_s = \sum_s b^s_s = \mathrm{tr}(B)$$

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Thus, the identity map and the trace function are the same tensor!

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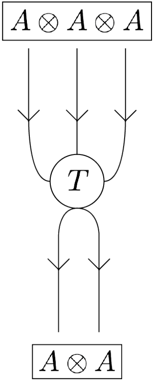
Wiring diagrams represent tensors and tensor operations.



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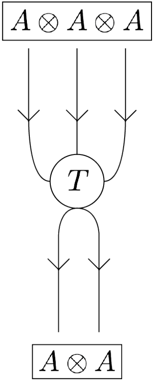
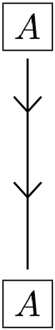


This tells us that $T : A \otimes A \otimes A \to A \otimes A$. That is, $T \in (A^*)^{\otimes 3} \otimes A^{\otimes 2}$.

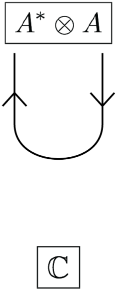
The identity in wiring diagrams

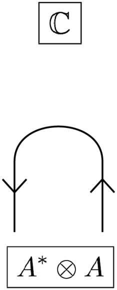
Here is the simplest representation of the identity as a wiring diagram:

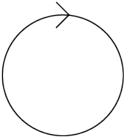


$$Id_A : A \to A$$

The identity in wiring diagrams







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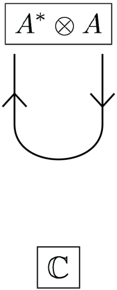

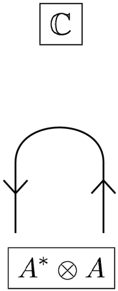

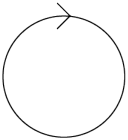

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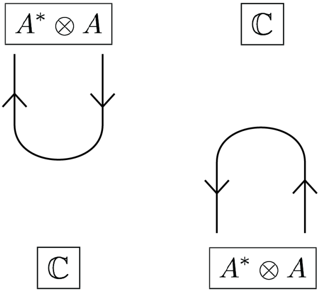

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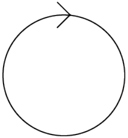

Tensor operations are preserved, so the image on the right can be obtained by composing the two images on the left...
$\lambda \mapsto \lambda Id \mapsto \lambda tr(Id) = \lambda \dim(A) \implies$ the rightmost picture represents $\dim(A)$.

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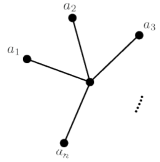
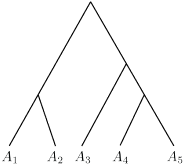
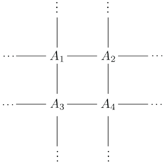
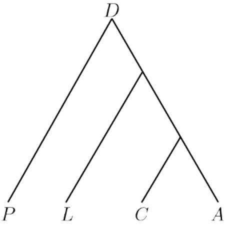
A brief look at some applications

Algebraic statistics	
Phylogenetics	
Physics	

Back to the beginning...



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More formally, we check,

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References and Thanks

Landsberg, J. M.
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Wiring diagrams in §2.11*; tensor networks in* §14.1.
American Mathematical Society, 2011.

Eisert, J.
Entanglement and Tensor Network States
Freie Universität Berlin, 2013.

Huge thanks to Dr. Joseph M. Landsberg and Dr. Philip Yasskin, who advised me on much of this talk and research.

What is a tensor network?

Given a graph, Γ, with m edges, e_s, and n vertices v_i, where every edge as a weight $\mathbf{e}_s$, and given vector spaces $V_1, \cdots, V_n$ corresponding to the vertices, we can construct a tensor network state.

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To be clearer, $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ denotes the tensor network state corresponding to the graph, Γ, where $\vec{\mathbf{e}} = (\mathbf{e}_1, \cdots, \mathbf{e}_n)$ and $V = V_1 \otimes \cdots \otimes V_n$.

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Constructing a tensor network state

1. Associate V_j to the vertex v_j and an auxillary vector space $E_s \cong \mathbb{C}^{\mathbf{e}_s}$ to the edge e_s.

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3. Let $\mathbf{V} = V_1 \otimes \cdots \otimes V_n$, and say that $s \in in(j)$ means that e_s is an incoming edge to v_j and $s \in out(j)$ means e_s is an outgoing edge.

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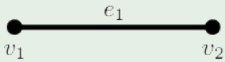
4. Define:

$$TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V}) := \{T \in V_1 \otimes \cdots \otimes V_n : \exists$$
$$T_j \in V_j \otimes (\otimes_{s \in in(j)} E_s) \otimes (\otimes_{t \in out(j)} E_t^*)$$
$$\text{such that } T = \mathrm{Con}(T_1 \otimes \cdots \otimes T_n)\}$$

What is a tensor network?

Examples

If Γ is:



then $TNS(\Gamma, \mathbf{e}_1, V_1 \otimes V_2)$ is the set of elements of $V_1 \otimes V_2$ of rank at most $\mathbf{e}_1$.

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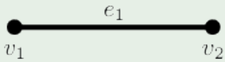

then $TNS(\Gamma, \mathbf{e}_1, V_1 \otimes V_2)$ is the set of elements of $V_1 \otimes V_2$ of rank at most $\mathbf{e}_1$.

<u>Construction</u>: WLOG, pick e_1 going in to v_1. Let $t_1, \cdots, t_{\mathbf{e}_1}$ be a basis for E_1 and $\tau^1, \cdots, \tau^{\mathbf{e}_1}$ be a basis for E_1^*. Then, a general tensor $T_1 \in V_1 \otimes E_1$ can be written as $T_1 = v_1 \otimes t_1 + \cdots + v_{\mathbf{e}_1} \otimes t_{\mathbf{e}_1}$. Similarly, a general tensor $T_2 \in V_2 \otimes E_1^*$ can be written as $T_2 = w_1 \otimes \tau^1 + \cdots w_{\mathbf{e}_1} \otimes \tau^{\mathbf{e}_1}$. The contraction of these tensors is $v_1 \otimes w_1 + \cdots + v_{\mathbf{e}_1} \otimes w_{\mathbf{e}_1}$, which is of rank at most $\mathbf{e}_1$.